GEOMETRICAL TRIGONOMETRY.

OR

The explanation of such

GEOMETRICAL PROBLEMS

As are most useful & necessary, either for the construction of the

CANONS of TRIANGLES. Or for the solution of them.

Together with

The Proportions themselves suteable unto every Case both in plain and Spherical *Triangles*, those in Spherical being deduced from the Lord *Nepeirs* Catholick or Universal proposition.

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Mathematical Sciences in their full latitude doe comprehend all Arts and Sciences whatsoever, but in a more strict sense, those onely are esteemed such, which doe belong to the Doctrine of Quantity, & amongst these this of Triangles is not of least concernment, of which though many have written and that excellently, yet still there may be something added, at least by way of explanation, if not some matter absolutely new. What we

have here pupliſhed is intended for beginners onely, in which we have endeavored after brevity & perſpicuity both, in the conſtruction of the Canon of natural ſines, Tangents and Secants firſt, & then the Axioms, & Problems for the ſolution of all Triangles, both plain & Spherical, either by thoſe natural numbers, or by the Logarith. of them, & in the ſolution of plain Triangles by logarithmetical numbers, ſometimes the logarithmes of a Decimal fraction is required, for

for the finding whereof having given no direction in our Preface to those Tables, we have (though somewhat out of place) inserted it her.

A fraction whether Vulgar or Decimal being given to find the logarithme thereof.

That the way & method here propounded may be the better conceived, the proper characteristique to the logars of any integer or whole number must be considered, and the characteristique of the logarithmes of all numbers, under 10, *is* 0, *of all numbers between* 10 *and* 100 *is* 1, *between* 100 *&* 1000 *is* 2, *& so forward, nor will there, as I conceive, be any inconvenience, if* 10 *be the characteristique of the logarithme of any digit, &* 11 *the characteristique of all numbers between* 10 *&* 100, *&* 12 *the characteristique of all numbers between* 100 *&* 1000, *& so forward; & yet if this shall be supposed, the characteristique of a number that is but one place beneath unitie shall be* 9, *if it be two*

places

places beneath unitie it shall be, &c.

Thus the Log. of 5 will be	10.698970
The Log. of 5 tenths will be	9.698970
The Log. of 05 hundreds will be	8.698970

And upon this ground to find the Log. of a vulgar fraction, you must subtract the logarithme of the denominator from the logarithme of the numerator, and what remaineth shall be the logarithme of the vulgar fraction given.

Example, Let $\frac{6}{12}$ *be the vulgar fraction given.*

Log.	*6 the Numerator is*	0.778151
	12 the Denominator is	1.079181
There difference		9.698970

is the log. of $\frac{6}{12}$ *the fraction propounded.*

And thus the logarithme of any fraction, or the decimal fraction answering to any logarithme, may be as easily found, as by any directions hitherto given, and in their use are much more ready, as will be manifest by the Example following.

If 6 tenths of an ounce of Gold cost 95 hundreds of a pound sterling, what shall 07 hundreds of an ounce of the same gold cost. *Answer is* 1108.

Log.

Logar. 6 tenths is	9.778151
Logar. 95 hundreds is	9.977724
Logar. 07 hundreds is	8.845098
	1.822822
Answer 1108	9 044671

To resolve this question, you must adde the Logarithmes of the second & third terms together, & from their sum subtract the first, what remaineth will be the logar. of the fourth proportional required, the characteristique of which fourth proportional doth by inspection shew, whether it be the logarithme of a fraction or not, there is but one difficulty in this way of working which is to substract a greater number from a less, in which if you will but borrow from a supposed figure as we doe from a real, and set down the difference that difficultie is also avoided, as in the first Example. Though I can substract 1 out of 1 yet I cannot substract 8 out of 5, but borrowing one from the next figure, I can substract 8 out of 15 and there will remain 7, and so proceeding till you come the characteristique the

remainer will be found in the ordinary way, but the characteristigne of the upper number being a cypher, and that of the lower an unite I cannot subtract 1 from 0, but supposing an unite to be beyond the cypher I can subtract 1 from 10 and there will remainer 9, the like is to be done in other cases, which I must leave to the consideration of the Ingenious Reader, *here being no room for to explain my self.*

JOHN NEWTON.

An explication of the Symbols.

= Equal to
+ More
— Less
x Drawn into
:: Proportionality
Z The Sum
X The Difference
s Sine
cs Co-sine
t Tangent
ct Co-tangent
R. ang. Right angle
2 R. ang. Two right angles.
q. Square.

CHAP. I.

Definitions Geometrical.

OF things *Mathematical* there are two principal kinds *Number*, and *Magnitude*, and each of these hath his proper science.

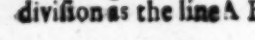
2 The science of Number is *Arithmetick*, and the science of Magnitude is commonly called *Geometry*; but may more properly be termed *Magethelogia* as comprehending all Magnitudes whatsoever, whereas *Geometry*, by the very Etymologie of the word doth seeme to confine, this science to *Land measuring* onely.

3 Of this Magethelogia, Geometry or science of Magnitudes, we will set down such grounds and principles, as are necessary to be known, for the better understanding of that which followeth, presuming that the Reader hereof hath already gotten some competent knowledge in *Arithmetick*.

4 Concerning then this ſcience of Magnitudes, two things are to be conſidered. 1 The ſeveral heads to which all Magnitudes may be referred. 2 The terms and limits of thoſe Magnitudes.

5 All magnitudes are either lines planes, or ſolids, and doe participate of one or more of theſe dimenſions, length, breadth, and thickneſs.

6 A line is a ſuppoſed length, or a thing extending it ſelf in length, without breadth, or thickneſs: whether it be, a right line or a crooked, and may be divided into parts in reſpect ofts length, but admitteth no other diviſion as the line A B.



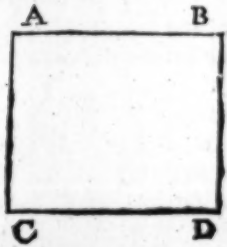
7 The ends or limits of a line are points as having his beginning from a point, and ending in a point; and therefore a point hath neither part nor quantity. As the points at A & B

are

are the ends of the afore-saidline A B, and no parts thereof.

8 A plain or superficies is the second kind of magnitude, to which belongeth two dimensions length, and breadth, but not thickness.

As the end, limits, or bounds of a line are points confining the line: So lines are the limits bounds and ends inclosing a superficies, as in the figure A B C D, the plain or superficies thereof is inclosed with the four lines A B, B D, D C, C A, which are the extreams or limits thereof.



9 A body or solid is the third kind of magnitude, and hath three dimensions, length, breadth, and thickness.

And as a point *is the limit or term of a line, and a* line *the limit or term of a superficies; So likewise a* superficies *is the end and limit of a* body *or* solid, *and representeth to the eye the shape or figure thereof.*

10 A Figure is that which is contained under one or many limits, under one bound or limit, is comprehended a Circle, and all other figures under many.

11 A Circle is a figure contained under one round line, which is the circumference thereof: *Thus the round line* C B D E *is called the circumference of that Circle.*

12 The center of a Circle is the point which is in the middest thereof, from which point all right lines drawn to the circumference are equal to one another: *As in the following figure, the lines* AB, AC, *and* AD, *are equal.*

13 The Diameter of a Circle, is a right line drawn through the center thereof, and ending at the circumference on the other side, dividing the

circle

circle into two equal parts. As the lines C A D and B A E, are either of them the Diameter of the Circle C B D E, because that either of them doth pass through the Center A, and divideth the Circle into two equal parts.

14 The Semidiameter of a circle is halfe the Diameter, and is contained between the center, and one side of the Circle. As the lines A B, A C, A D, and A E, are either of them the Semidiameter of the Circle C B D E.

15 A Semicircle is the one halfe of

a circle drawn upon his Diameter, & is contained by the half circumference and the Diameter. *As the Semicircle C B D is half the Circle C B D E and drawn upon the Diameter C A D.*

16 A quadrant is the fourth part of a Circle, and is contained between the Semidiameter of the circle, and a line drawn perpendicular unto the Diameter of the same circle from the center thereof, dividing the Semicircle into two equal parts, of the which parts the one is the quadrant or fourth part of the circle. *Thus from the center A the perpendicular A B being raised perpendicularly upon the Diameter C A D, divideth the Semicircle C B D into the two equal parts C F B & F G D each of which is a Quadrant or fourth part of the circle C B D E.*

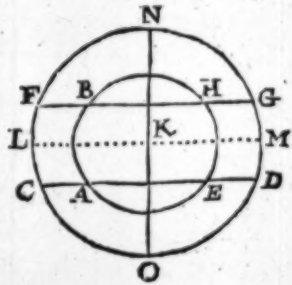
17 A Segment or portion of a circle, is a figure contained under a right line, and a part of the circumference of a circle, either greater or lesser then a Semicircle. *As in the former figure F B G H is a Segment or part of the circle*

circle CBDE, contained under the right line FHG, lesse then the Semicircle CBD.

And by the application of the severall lines or terms of a superficies one to another, are made parallels, angles, and many sided figures.

18 A parallel line, is a line drawn by the side of another line, in such sort that they may be equi-distant in all places, and of such there are two sorts, the right lined parallel, and the circular parallel.

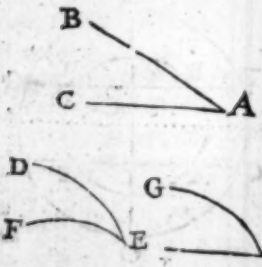
19 Right lined parallels are two,



right lines equidistant in all places one from another, which being drawn to an infinite length, would never meet or concur, *As the right lines F H G & C A D in the annexed Diagram.*

20 A Circular parallel is a circle drawn within or without another circle, upon the same center, as you may plainly see by the two circles CFGD, and A B H E which are both drawn upon the same center K, & are therefore parallel to one another.

21 An angle is the meeting of two lines in any sort, so as they both make not one line: as the two lines A B



and

nd A C incline the one to the other nd touch one another in the point A, in which point is made the angle B A C. *Where note, that an angle is for the most part described by three letters of which the second or middle letter representeth the angular point. Thus in the angle B A C the letter A representeth the angular point.*

If the lines which containeth the angle be right lines, it is called a right lined angle: *As the angle B A C.*

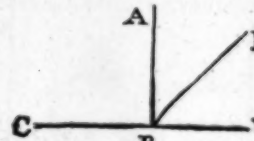
A crooked line angle, is that which is contained of crooked lines, *as the angle D E F.*

And a mixt angle, is that which is contained both of a right and a crooked line, *as the angle G H I.*

22 All angles are either right or oblique.

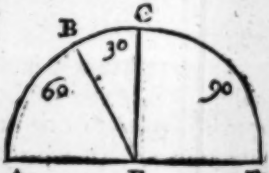
23 A right angle is an angle contained between two right lines, drawn perpendicular to one another. *Thus the angle A B C is a right angle, because the right line A B is perpendicular to the right line C D and the contrary-*

24 An oblique angle, is an angle contained between to right lines not perpendicular to one another: *as the angle C B E or E B D, the one whereof*



[viz.] *E B D is acute or less then a right angle, and the other* [viz.] *C B E is obtuse or more then a right*, & are the complements of each other to a Semicircle.

25 The measure of an angle is the arch of a circle described on the angular point, and intercepted between the two sides of the angle.



Thus in the annexed Diagram, the arch A B is the measure of the angle

gle A E B, & that the quantity thereof may be the better known.

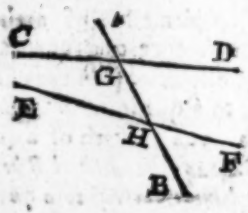
26 Every circle is supposed to be divided into 360 parts or deg. every degr. into 60 min. or 100 parts, &c. Therefore a Semicircle as the arch A C D is 180 deg. a quadrant or fourth part of a Circle, as the arch A B C is 90 deg.

27 Complements of arches are either in reference to a quadrant or a Semicircle, the complement of an arch or angle to a quadrant is so much as the arch given wanteth of a quadrant or 90 deg. *as if the arch A B be 60 deg. the complement thereof to a quadrant is the arch B C 30.*

In like manner, the complement of an arch or angle to a Semicircle, is so much as the arch or angle given wanteth of a Semicircle; as if the arch B E D be 120 degrees, the complement thereof is the arch A B 60 deg.

28 When a right line falling upon two right lines doth make on one and the same side, and two inward angles

less the two right, the right lines being produced will at length concur in that part in which the two inward angles are less then two right. As in the figure, the right line A B falling upon the right lines C D and E F maketh the angles C G H and G H E together less



then two right angles, if therefore the lines C D and E F be produced from C and E they will at last concur.

29 Many sided figures are such as are made of three, four, or more lines, though for distinction sake those onely are so called which are contained under five lines or terms at the least.

In

In this Treatise we have to do with such onely as are contained under three lines or sides, & these are therefore called Triangles, for the better understanding whereof we will here set down some necessary and fundamental Propositions of *Geometry*.

CHAP. II.

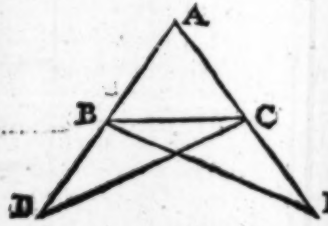
Propositions Geometrical.

1. PROP. *If two sides of one Triangle be equal to two sides of another, and the angle comprehended by the equal sides equal, the third side or base of the one shall be equal to the base of the other, and the remaining angles of the one, equal to the remaining angles of the other.*

Demonst. In the Triangles C B H & F E D make C B = F E and B H = E D, and

and ang. CB=FED, then shall CH =FD, For if CG=FD, the angle CBG=FED, but CBH=FED by construction, therefore CH=FD, ang. C=F, and the angle H=D as was to be proved.

2 Prop. *If a Triangle have two equal sides the angles at the base are also equal to one another, and the contrary.*

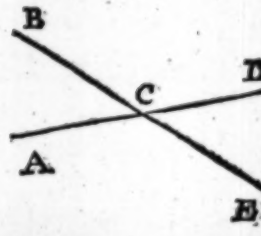


Demonst. In the Triangle A B C let A B = A C, and let A B be extended to D and A E = A D, and draw the lines B E and D C, now then because A D = A E and A B = A C and the angle A common to the Triangle A D C and A B E, the base B E = D C the angle D = E and A B E = ang. A C D by the former prop. and therefore

fore ang. D C B = E B C and being taken from the equal angles A B E & A C D, there shall remain the angle A B C = A C B as was to be proved.

3 Prop. *If two right lines doe cut through one another, the angles opposite to one another are equal.*

Demonst. Ang. A C B + B C D = 2 right ang. 24 of the first, ang. A C B + A C E = 2 right, and ang. A B C common, therefore ang. B C D = A C E as was to be proved.



4 *If the side of a Triangle be extended the outward angle shall be greater than any one of the inward opposite angles.*

Demonst. In the Triangle A B C

les

let A C be biſected in F and F G = F B, then is ang. A F G = B F C by the 3 prop. and B C = A G by the firſt thereof, as alſo, ang. G A F = F C B and therefore ang. B A C greater then A C B as was to be proved.

5 Prop. *In all Triangles, the greateſt ſides ſubtend the greateſt angles, and the leſſer ſides, the leſſer angles.*

Demonſt. In the Triangle A B C if B D = B C and draw D C, then ang. B D C = B C D, & ang. B D C greater then D A C by the 4th. here of, therefore ang. A C B greater then B A C as was to be proved.

6 *If a right line drawn through two other right lines, doe make the alternate angles equal, then are those two right lines parallel.*

Demonst. Let ang. AGH=GHD and then if AB and CD be not parallel they will at length concurre

(suppose at K) by the 28 of the first, & the angle GHD greater then AGH by the fourth hereof, the which is contrary to the proposition, if therefore the alternate angles be equal, the lines being extended shall not concur, and are therefore parallel, by the 19th of the first, as was to be proved.

7 *If a right line drawn through two other right lines, shall make the outward angle of the one, equal to the inward angle of the other on the same side, or both the inward angles equal to two right, then are those two right lines parallel.*

Demonst. Ang. AGE = BGH by the

the third hereof, and ang. AGE = GHC by the propoſition, therefore BGH = GHC and AB parallel to CD by the laſt afore-going.

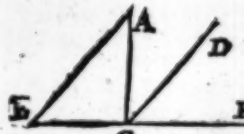
Again, ang. AGE + AGH = 2 R. ang. by 24 firſt, and AGH + GHC = 2 R. ang. by the propoſition, therefore AGE = GHC & AB parallel to CD as before.

28 *If one right line cut through two parallel right lines, the angles oppoſite to one another are equal.*

Demonſt. Ang. AGH + BGH = 2 R. ang. and ang. CHG + GHD = 2 R. ang. by the 24 of the firſt, therefore, ang. CHG = BGH, for if BGH be greates CHG ang. CHG + AGH — 2 R. ang. and AB not parallel to CD by the 28 of the firſt, which is contrary to the propoſition, and therefore AB being parallel to CD, ang. AGH = GHD as was to be proved.

19 *If the side of a Triangle be continued, the outward angle made by the continuation, is equal to the inward opposite angles, and three inward angles are together equal to two right.*

Demonst. Let C D be parallel to A B then shall ang. B A C = A C D and ang. A B C = D C E by the last afore-going, and therefore ang. A C E = A B C + B A C.



Again, ang. A C B + A C E = 2 R. by the 24th of the first, and the angle A C E = A B C + B A C, therefore ang. A C B + B A C + A B C = 2 R. ang. as was to be proved.

10. *If two equal parallel lines be joyned together with two other right lines, those other lines are equal and also parallel.*

Demonst. Ang. V X W = X W Y and V X parallel W Y by the proposition, and X W common to both, there-

therefore V W = X Y by the firſt hereof, and alſo parallel by the 7th hereof, as was to be proved.

11 *In all Paralellograms, the angles and ſides that are oppoſite to one another are equal, and the Diameter or Diagonal divideth the ſame into two equal parts.*

Demonſt. In the preceeding Diagram ang. V W X = Y X W, and angle X W Y = V X W by the 8th hereof, and X W common to both the Triangles V X W and Y X W, therefore the other ang. and ſides are alſo equal by the firſt hereof, and the paralellogram V Y is divided into two equal parts by the diagonal X W as was to be proved.

12 *All paralellograms and Triangles ſtanding upon the ſame or equal baſe, and between two parallel lines (that is having the ſame common height) are equal.*

Demonſt. The two paralellograms

NL and NO ſtand upon the ſame baſe KN, and between the two paralel lines KR and LP. The ſide KL = NM and KO = NP by the 11th hereof: and the Triangle KOL = NPM. From theſe two take the

common Trian. MIO & there will remain the quadrilateral figure LKIM = NIOP to which if you adde the common Triangle KIN the paralellogram LN = NO.

Again upon the equal baſes KN & QR do ſtand the paralellograms LN & PQ which let be connected by the lines KO and NP being parallel by the 10th hereof, and include the paralellogram NO = NL as was proved before, and may in like manner be proved to be equal to QP, there-

therefore NL = QP, and what is said of Paralellograms is also true of Triangles they being the halves of paralellograms.

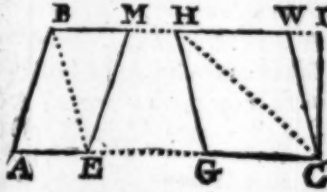
COROLLARY.

In a rectangular paralellogram, the Area is found by multiplying the unequal sides together, and any Obliquangular paralellogram is equal to a Rectangular one made upon the same base and altitude, the Area thereof is therefore found by multiplying the base by its altitude which is the other side of the Rectangular paralellogram, the same is to be understood of there halves being Triangles.

13. *Paralellograms and Triangles having the same common height are in proportion as there bases.*

Demonst. Let the paralellograms A M and G W stand between the parallel lines B W and A C, and have for there equal and common height the perpendicular P C and by the corallary

rallary of the 12th hereof C P × AE = A *M* and P C × G C = G W, therefore P C × A E, P C × G C :: A E, G C, and also A M, G W, :: A E, G C, as was to be proved.



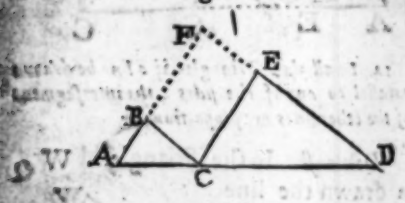
14. *In all plane Triangles, if a line be drawn parallel to any of the sides, the interfegment of the other sides are proportional.*

Demonſt. In the Triangle M W Q is drawn the line I H parallel to M Q, and the Triangles I H M = H I Q by the 12th hereof, and by the laſt afore-going, Triangle W I H. H I Q. :: W H, H Q. Triangle W I H. H I Q. :: W I. I M, therefore W H,

WHHQ: : WI, IM as was to be demonstrated.

15 *Equiangled Triangles have there sides which contain or subtend equal Angles proportional, and the contrary.*

Dem. Let the Triangles A B C and C E D be equiangled, and let their bases A C and C D make one line, & extend the sides A B and D E till they meet in F, now because ang. B A C = E C D and ang. B C A = E D C



the sides A B and C E are parallel by the 7th hereof, and also the sides B C and E D which being extended make the paralellogram C B F E in which B F = C E and B C = F E by the 11th hereof, it followeth, by the last afore-going,

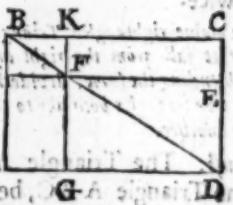
A B

AB. BF(=CE) :: AC, CD, and also AC, CD :: EF, DE.

And DC, CA :: BF, AB as was to be proved.

16. In all paralellograms the supplements or fillings which stand upon the Diameter are equal.

Demonst. The Diameter or diagonal BD bi-secteth the paralellograms AC, HK and GE by the 10th hereof, and the Triangle ABD = BCD from which taking Triang. FGD = FED as also the Triang. BHF =



GKF there will remaine the paralelgram HG = FC as was to be proved.

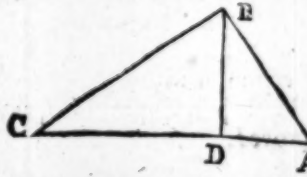
17 *If four right lines be proportional, the Rectangled figure made of the two meanes, is equal to the Rectangled figure made of the two extreams.*

Demonst. In the preceding Diagram, the Triangles G F D and B H F are like, because the ang. B F H = F D G and H = G by the 7th hereof, and therefore B H. H F :: F G. G D by the 15th hereof, & the Rectangle F C = B H × G D and the rectangle H G = H F × F G by the last aforegoing as was to be proved. The like may be said of three proportionals taking the meane twice.

18 *In a plaine rightangled Triangle a perpendicular let fall from the right angle upon the side subtending the same, divideth the Triangle into two triangles both like to the whole, and to one another.*

Demonst. The Triangle A B D is like to the Triangle A B C, because B and D are right, and A common to both: and the Triangles A B C and B D C are like, because B and D are right & C common to both, and therefore the Triangles B C D and B A D are

are like, as was to be proved.



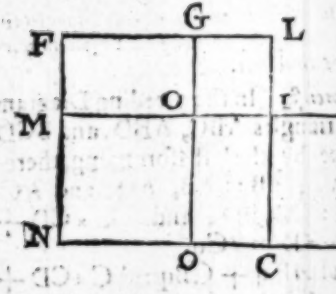
19 *In all right angled plaine Triangles the sides including the right angle are equal in power to the third side.*

Demonst. In the preceding Diagram the Triangles ABC, ABD, and BCD are like, by the last aforegoing, therefore AC, AB :: AB, AD, and AC, CB, :: CB, BD, and AC × CD + AC × AD = ACq.

And ABq + CBq = AC × CD + AC × AD, therefore ABq + CBq = ACq, as was to be proved.

20 *If a right line being divided into two equal parts shall be continued at pleasure, the rightangled figure made of the line continued, and the line of continuation, with the square of one of the bi-segments, is equal to a square made of one of the bi-segments, and the line of continuation.*

Demonst. Let P Q be biſected in C and continued to N and let Q C = O I = L I then the ſquare O L + M P = N C q for F O = O C or I P and M C common to both, therefore N P × M N + Q C q = N C q as was to be proved.



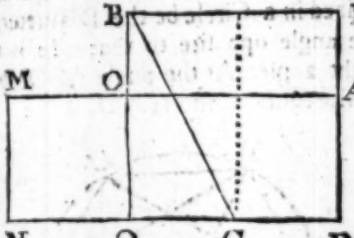
11 *To divide a right line given by extream and meane proportion, that is, that the right angled figures made of the whole line and one part ſhall be equal to the ſquare of the other part.*

Demonſt. If the right line Q P = Q B be bi-ſected in C and a line drawn from C to B, and the line N C = C B

and

and MN = N2, the rectangled figure made of B 2 × 3 O = O 2 q for by the last aforegoing M P + 2 C q = N C q, or B C q. And B 2 q + 2 C q = B C q by the 19 hereof.

Therefore B 2 q = N P × M N &



O P being common to M P and B P, it followeth that M 2 = B A as was to be proved.

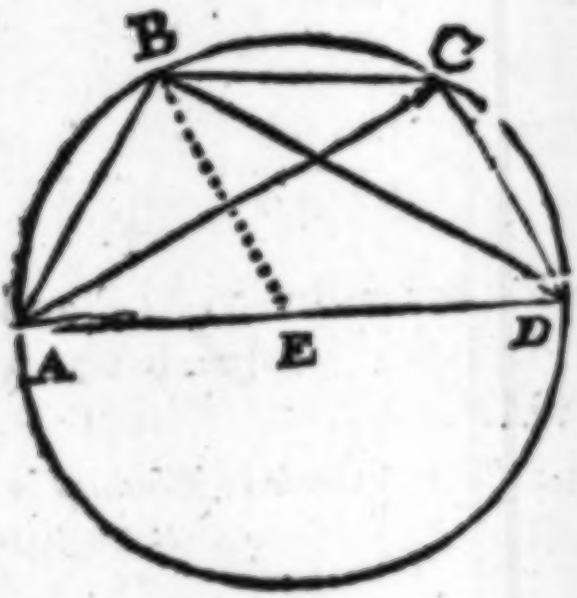
22 *If a plaine Triangle be inscribed in a circle the angles opposite to the circumference, are half as much as that part of the circumference which is opposite to the angles.*

Demonst. In the Triangle EBD, ang. EDB = EBD by the second hereof, and ang. A E B equal to both by the 9th hereof, the arch A B is the mea-

ſure of the angle A E B, by the 25th of the firſt, therefore the arch A B is the double meaſure of the angle ADB as was to be proved.

1 *Conſectary.*

If the ſide of a plaine Triangle inſcribed in a Circle be the Diameter, the angle oppoſite to that ſide is a right angle. As the angle ABD oppoſite to the diameter A D.



2 *Conſectary.*

If divers right lined Triangles, be inſcribed in the ſame ſegment of a circle upon one baſe, the angles in the cir-

circumference are equal. As the Triang. ABD & ACD being inſcribed in the ſame ſegment of the circle ABCD and upon the ſame baſe AD, have their angles at B and D falling in the circumference equal.

23 *If a quadrilateral figure be inſcribed in a circle, the angles thereof which are oppoſite to one another, are together equal to two right angles.*

Demonſt. Ang. CDB=CAB & BDA=BCA by the laſt aforegoing, therefore ang. CDA=BCA+BAC and ABC+BAC+BCA=2 R. ang. by the 9th hereof, & therefore ang. ABC+ADC=2 R. ang. as was to be proved.

24 *If in a quadrilateral figure inſcribed in a circle, there be drawn two Diagonal lines, the rectangle under the Diagonals, is equal to the two rectangles under the oppoſite ſides:*

Demonſt. Let ang. DAE=CAB by conſtruction, then ſhall ang. DAC=EAB and ang. ACD=ABE becauſe the arch AD is the double meaſure to them both. and therefore the triangles ADC & AEB are like.

Again

Again ang. A D B = A C B because the arch A B is the double measure to them both and ang. D A E = C A B by construction, & the Triang. A E D and A B C like, therefore, A C, C B :: A D, D E.

And A C, C D :: A B, B E.
Therefore A C × D E = C B × A D.
And also A C × B E = C D × A B
And A C × D E + A C × B E = A C × D B. Therefore A C × D B = C B × A D + C D × A B as was to be proved.

Chap.

CHAP. III.

Of the Construction of the Canon of Triangles.

THat the Proportions which the parts of a Triangle have one to another may be certain, the arches of circles (by which the angles of all Triangles, and of Spherical Triangles the sides are also measured) must be first reduced into right lines, by defining the quantity of right lines, as they are applyed to the arches of a circle.

2 Right lines are applyed to the arches of a circle three wayes, *viz.* either as they are drawn within the circle, without the circle, or as they are drawn through it.

3 Right lines within the circle are Chords and sines.

4 A Chord or subtense is a right line inscribed in a circle, dividing the whole circle into two segments: and in like manner subtending both the

segments: as the right line C K divideth the circle G E D K into the two segments C E G K and C D K, and subtendeth both the segments, that is, the right line C K is the chord of the arch C G K, and also the chord of the arch C D K.

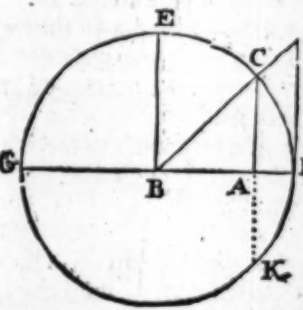
5 A Sine is a right line in a semicircle falling perpendicular from the term of an arch.

6 A Sine is either right or versed.

7 A right Sine is a right line in a Semicircle, which from the term of an arch is perpendicular to the diameter, dividing the Semicircle into two segments, and in like manner referred to both: Thus the right line C A is the sine of the arch C D less then a quadrant, and also the sine of the arch C E G greater then a quadrant, and hence instead of the obtuse angle G B C, we take the acute angle C B A the complement thereof to a Semicircle, and so our Canon of Triangles doth never exceed 90 deg.

8 *A* right ſine is either *Sinus totus*, that is, the Radius or whole Sine, as the right line E B: or *Sinus ſimpliciter* the firſt ſine, or a ſine leſs then Radius, as A C or *A* B, the one whereof is alwayes the complement of the other to 90 degrees, we uſually call them ſine and co-ſine.

9 A Verſed ſine is a right line in a Semi-circle perpendicular to the right



Sine of the ſame arch; Thus A D is the verſed ſine of the arch C D and G B A is the verſed ſine of the arch

G E C the complement of C D.

10 Right lines without the Circle whose quantity we are to define, are such as touch the circle, and are called Tangents.

11 A Tangent is a right line which touching the circle without, is perpendicular from the end of the diameter to the Radius, continued through the term of that arch of which it is the Tangent: Thus the right line F D is the Tangent of the arch C D.

12 Right lines drawn through the circle, whose quantity we are to define are such as cut the circle and are called Secants.

13 The Secant of an arch, is a right line drawn through the term of an arch, to the Tangent line of the same arch: and thus the right line B F is the Secant of the arch C D: as also of the arch C E G the complement thereof to a Semicircle.

14 A Canon of Triangles then is that which conteineth the Sines, Tangents, and Secants of all degrees & parts

parts of degrees in a quadrant, according to a certain diameter, or measure of a circle assumed: The construction whereof followeth, and first of the Sines.

15 The right Sines as they are to be considered in order to their construction are either Primary or Secondary:

16 The Primary Sines are those by which the rest are found: *And thus the Radius or whole sine is the first primary sine, and is equal to the side of a six-angled figure inscribed in a Circle.*

Demonst. AC=AB by the work,

and

and ang. C = B by the second of the second hereof, and B C the measure of the angle at A is 60 degr. and the angles A, B, C, are together equal to two right angles, and therefore ang. A = C or B and BC = AC or AB as was to be proved.

Corollary.

The Radius of a circle being given, the sine of 30 deg. is also given, for by this proposition, the Radius of a circle is the subtense of 60 deg. and the half thereof is the sine of 30, and therefore the Radius A B, or B C being 1000. 0000 the sine of 30 deg. is 500. 0000.

17 The other primary sines are the sines of 60. 18 and 12 deg. being the half of the subtenses of 120. 36 & 24 degr. and may be found by the problems following.

18 *The right sine of an arch & the right sine of its complement, are in power equal to Radius.*

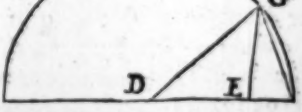
Demonst. In the first diagram of this chapter, A C is the sine of C D, and

and A B the sine of C E the complement thereof, which with the Radius B C make the right-angled Triangle A B C, therefore A B q + A C q = B C q by the 19 of the second, as was to be proved.

And hence the sine of 60 deg. may thus be found, let the sine of 30 deg. A C be 500. 0000 the square whereof 250. 00000 being subtracted from the square of B C Radius, the remainer is 750. 00000 the square of A B, whose square root is 866254 the sine of 60 deg.

19 *The subtense of 36 deg. is the side of a Dec-angle inscribed in a circle, or the greater segment of a Hexagon divided into extream and mean proportion.*

Demonst. Let D G = D A and E G



bisect the ang. A G D = D A G by the second

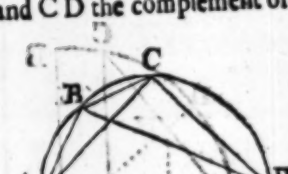
second of the second, and the angle DGE=ADG because AGD half the complement of ADG(36 deg.)to a Semicircle is bi-sected by the right line EG=DE by the second of the second, and the angle AEG=EGD +EDG,and therefore AEG=EAG =AGD and EG=AG by the second of the second,and the Triangles ADG and EAG like, therefore AD, AG=ED :: AG AE,and so AD the side of a Hexagon is divided into extream and meane propo tion, & ED the greater segment equal to AG the side of a dec-angle, as was to be proved.

Corollary.

The side of a Hexagon being given, the side of a Dec-angle or subtense of 36 degr. is also given, for by the second of the second the Semi-radius being deducted from the square root of the squares of Radius, and the half Radius added together, the remainer is 6180339 the subtense of 36 deg.

20 *The subtense of 24 deg. is the side of a Quin-decangle, or the difference between the subtenses of 60 and 36 deg. and may be found by the 24 of the second.*

Demonst. Let A C be the subtense of 60 deg. A B 36, then is B C 24 the difference, B D the complement of A B and C D the complement of A C



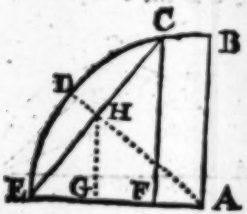
and A D the diameter. And A C × BD = AB × C D + BC × A D therefore A C × B D − A B × C D = A D × B C, and A D × B C being divided by A D the diameter, the quotient is BC the subtense of 24 deg.

21 *Having thus found the primary sines, the secondary sines as the sines of 6 d. g 3 deg. 1 d. 50 cent. 0 deg. 75 centesmes, and 0 d. 01 centesme may be found from them, and all the other sines, to every degree, and part of a degree in the quadrant.*

22 *The*

22 *The right sine and versed sine of an arch are together equal in power to the subtense of the same arch.*

Demonst. The right sine C F and the versed sine F E with the subtense E C doe make a right angled Trian-



gle, and therefore F C q + F E q = E C q by the 19 of the second, as was to be proved,

Corollary.

The right sine and versed sine of an arch being given, the sine of half that arch is also given, for E H the $\frac{1}{2}$ E C is the sine of E D the halfe of the arch E D C.

23 *The*

23 *The right sine of an arch is a meane proportional betweene the Semi-radius, and the verſed sine of the double arch.*

Demonſt. In the preceding diagram the Triangles E H A and E C F are like, becauſe ang. F = E H A and E common to both, therefore A E. E C :: E H, E F or A E. E C :: $\frac{1}{2}$ A E : $\frac{1}{2}$ E C that is, $\frac{1}{2}$ A E, $\frac{1}{2}$ E C = E H :: E H, E F as was to be proved.

1 *Conſectary.*

Therefore by this or the former propoſition, the sine and sine complement of an arch being given, the sine of half that arch is alſo given, I ſay the sine complement, becauſe the verſed sine is found, by deducting it from Radius; Thus F A the sine of B C being deducted from A E Radius the remainer is F E the verſed sine of E D C and $\frac{1}{2}$ A E × E F = E H q whoſe root is E H or the sine of E D.

24 *The rectangle of the sine and sine complement of an arch is equal to the rectangle of half the Radius, and the sine of the double arch.*

Demonst. In the preceeding diagram, E H is the sine of E D and H A the cosine thereof, and C F is the sine of E D C the double arch, the Triangle H A E and E F C are like, as before.

And A E, E C :: H A, C F
And $\frac{1}{2}$ A E, E H = $\frac{1}{2}$ E C :: H A C E
And the rectangle of A H × H E = $\frac{1}{2}$ A E × C F as was to be proved.

25 *By these Propositions the sine of 12 deg. being given the sines and sines complements of these arches 6 deg. 3 deg. 1 deg. 50 cent. 0 deg. 75 cent. &c. were found to be as followeth.*

Deg. parts	Sines	Co-sines.
6.00	10452.84612	99452.18953
3.00	5233.59562	99862.95347
1.50	2617.69183	99965.73249
0.75	1308.95955	99991.43175
0.375	654.49379	99997.85816
0.1875	327.24865	99999.46453

And

And from the sine of 0 deg. 1875 parts, the sine of 10 centesmes may be found, in this manner.

As, 0 deg. 1875
Is to the sine thereof 327.24865
So is 0 deg. 10 centesmes.
To the sine thereof 174.53261
Therefore the sine of 0 d. 5 cent. 87.26930
The sine of 0 d. 01 cent. is 17.43326
And the Co-sine thereof is 99999.99847

which being given the rest of the Table of sines may be easily made by this proposition following.

26 Three equi-different arches being propounded, the rectangle made of the cosine of the common difference and the double sine of the meane arch, is equal to the Rectangle made of the Radius, and the summe of the sines of the two extream arches.

Demonst. Let the three equi-different arches be ED, EC, & EB, whose common difference is BC or CD the arch EF = DE, therefore the arch BEF = 2 CE, and BF = 2 CL the sine of

of GE and GF = BD the double measure of the ang. GBF = BAC and the ang. BNA & BHF right, therefore the Triang. ABN and BFH like AB, AN :: BF, BH the sum of BM and BK the sines of the extream arches, therefore AN × BF = AB × BH which was to be proved.

Corollary.

The sines of ED, and EC with the co-sine of CD or BC being given, the sine of EB is also given for if from BH the sum of BM and DK you deduct MH = DK the remainer is BM the sine of BE, for illustration sakewe have added these examples following.

1 Ex-

1 Example.

E D 5 cent. and E C 10 centesmes being given to find B M the sine of E B 15 centesmes.

As the Radius A B	10000.0000
To A N the cos. B C 5′ viz.	9999.9962
So B F the doub. of L C 10′	349.0650
To B H = B M + D K	349.0659
Fro which subt. M H = D K 5′ cent.	87.2663
Resteth the sine of od 15 cent. B M	261.7987

2 Example.

E D 0 deg. 10 cent.
E C 0 deg. 15 cent. given, to find E B 0 deg. 20

As the Radius A B	10000.0000
To cos. B C 5 cent. viz. A N	9999.9962
So the double of L C 15 cen. viz B F.	523.5974
To B H = B M + D K	523.5976
Subt. M H = D K 10 cent.	174.5326
Rest sine 0 deg. 20 cent. B M.	349.0650

3 Example.

Let E B 0 d. 25 cent. be inquired.

There being given

E D 0 deg. 15 cent.

E C 0 deg. 20 cent.

As

As the Radius AB 90	10000.0000
To Cos. BC viz. 1 AN 89 d 95	9999.9962
So the doub. of LC 1 o d 20 cent.	698.1300
To BH = BM + DK	698.1297
Sub. MH = DK o d. 15 cent.	261.7981
Rest BM the sine of o d. 25 cent.	436.3316

4 *Example.*

Let EB o d. 30 cent. be inquired.

There being given

E D o d. 20 cent.

E C o d. 25 cent.

As the Radius 90	1000.0000
To Cos. BC viz. 1 AN 89.95	9999.9962
So twice L' = BF viz. 25 cent.	872.6618
To B H = B M + D K	872.6613
Subt. D K = M H o d. 20 cent.	149.0950
Rest M B the sine of o d. 30 cent.	523.5963

And in this manner may the whole Canon of sines be compleated, or the sines of the first or last 60 deg. being thus made, or the sines of 45 deg. being made by this rule, and the sines of 15 d more by the 24th thereof, the other 30 d. may be more easily made by the following Problem.

26 *The ſum of the Sines of any two arches equally diſtant from 30deg. is equal to the co-ſine of the diſtance.*

Demonſt. In the preceding diagram we have already proved.
AB. AN : : BF. BH therefore alſo
½ AB. AN : : ½ BF. BH, now then if
½ B F = ½ A B ſine of 30 deg. A N the coſine of the diſtance muſt be alſo equal to B H, the ſum of the ſines, as was to be proved.

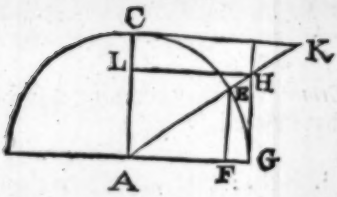
Example.

Let DK the ſine of DE 27 d. be 45399.04997
And BM the ſine of EB 33 d. be 54463.90350
Their ſum is BH = AN ſine of 87d. 99862.95347

27 *The Canon of ſines being thus made, a Table of Tangents and Secants may be eaſily deduced from them, by the following problemes.*

28 *As the co-ſine of an arch, is to the ſine thereof, ſo is Radius, to the Tangent of that arch.*

Demonſt. In the annexed diagram, the Triangles A E F and A H G are like, becauſe of their right angles at F and G, & their common angle at A. Therefore, AF. FE : : A G. GH.



29 *The Radius is a meane proportional, between the tangent and the tangent complement of an arch.*

Dem. HG is the tangent of an arch, CK the co-tang. thereof & LH=AG and the triangles, A L H and A C K are like, becauſe of their right angles at L and C, and their common angle at A. Therefore AL=HG. LH :: A C. CK.

30 *The Radius is a meane proportional, between the right ſine of an arch, and the ſecant of its complement.*

Demonſt. In the preceding diagram the triangles A E F & A H G are like, therefore, AF. AE :: AG. AH.

31 *As the ſine of an arch or angle is to Rad. ſo is the tangent of the ſame arch, to the ſecant thereof.*

Demonſt. In the preceding diagram the

the triangles AEF & AGH are like, therefore EF. AE :: HG. AH.

32 *As Radius, is to the secant of an arch, so is the co-tangent of the same arch, to the co-secant thereof.*

Demonst. In the preceding diagram, the triangles ALH and ACK are like, therefore LH, AH :: CK. AK.

Other more easie and expeditious wayes of making the Tangents and Secants, you may see in the first Chap. of my *Trigonometria Britannica*, but the Canons being now already made, these Rules we deeme sufficient.

The construction of the Artificial Sines and Tangents, we have purposely omitted, they being nothing els but the Logarithmes of the Natural, of which Logarithmes we have shewed the construction in a former Institution, by the extraction of roots, and in my *Trigonometria Britannica* by multiplication: and therefore shall now proceed to the use of the Canon of *Sines*, Tangents and *Secants*, in

the ſolution of all Triangles, whether plaine or Spherical.

Chap. IV.

Of the Calculation of plain Triangles.

A Plain Triangle is contained under three right lines, and is either Right-angled, or Oblique.

2 In all plain Triangles, two angles being given the third is alſo given: and one angle being given the ſum of the other two is given: becauſe the three angles together are equal to two right *by the 9th of the ſecond. Therefore in a plain right angled triang. one of the acute angles is the complement of the other.*

3 In the reſolution of plain Triangles, the angles onely being given, the ſides cannot be found, but onely the the reaſon of the ſides: It is therefore neceſſary, that one of the ſides be known.

4 In

4 In a Right-angled Triangle two termes (beſides the right-angle) will ſerve to find the third; ſo the one of them be a ſide.

5 In Oblique angled Triangles there muſt be three things given to find a fourth.

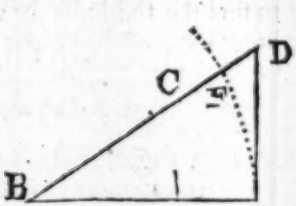
6 In right-angled plain Triangles there are ſeven caſes, and five in Oblique, for the ſolution of which the four Axiomes following are ſufficient.

1 *Axiome.*

In a right-angled plain Triangle: *The rectangle made of Radius & one of the ſides containing the right-angle, is equal to the rect-angle made of the other containing ſide, and the tangent of the angle thereunto adjacent.*

Dem. In the right angled plain triang. BED draw the periphery F E, then is BE Radius & DE the Tangent of the angle at B, make C A parallel to DE, then are the Triangles A B C and E B D like, becauſe of their right-angles at A and E and their common

angle at B, therefore BA. BE :: AC. ED. and BA × ED = BE × AC. that



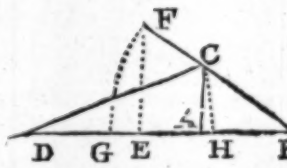
is BA × t B = Rad. × AC. as was to be proved.

2 *Axiome.*

In all plain Triangles : *The sides are proportional, to the sines of their opposite angles.*

Demonst. In the plain triangle BCD extend BC to F making BF = DC, and draw the arches FG & CH, then are the perpendiculars FE & CA the sines of the angles at B & D by the 7th of the third, and the triangles B E F and B A C are like, because of their right angles at E and A, and their common angle at B. Therefore BC.

C. C*A* : : BF. FE. that is



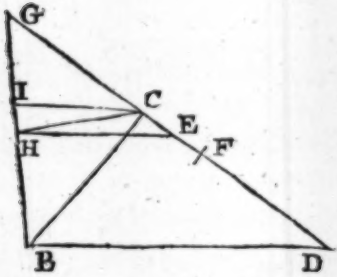
BC. sine D : : DC = *B*F. sine *B*. as was to be proved.

3 *Axiome.*

In all plain Triangles; *As the halfe sum of the sides, is to their halfe difference: so is the tangent of the half sum of their opposite angles, to the tangent of their half difference.*

Demonst. In the triangle *B* C D let the sides be C*B* and CD, and CG = C*B*. wherefore $\frac{1}{2}$ Z crur. = EG. & $\frac{1}{2}$ X crur. = E C. draw C H bi-secting *B* G at right angles, and make the angle G C I = D, then will the angle G C H = $\frac{1}{2}$ Z angle *B* and D whose tangent is H G, and the angle

ICH = $\frac{1}{2}$ X ang. *B* and D whose tangent is H I.



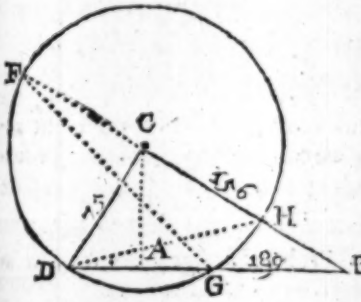
But E G. E C : : H G. H I. that is. $\frac{1}{2}$ Z crur. $\frac{1}{2}$ X crur. : : t $\frac{1}{2}$ Z ang t $\frac{1}{2}$ X ang.

4 *Axiome.*

In all plain triangles: *As the base, is to the sum of the other sides, so is the difference of those sides, to the difference of the segments of the base.*

Demonst. In the triangle *B*CD let fall the perpendicular AC, extend BC

C to F and draw F G and DH, then
BF = Z crur. and H B = X crur. &
GB the difference between A B and
AD the segments of D B the base.
And the triangles HDB and BGF are



like, because of their equal ang. at D &
F the arch HG being the double measure to them both, and their common angle at B. Therefore
DB. BF :: HB. GA. That is,
D B. Z crur. :: X crur. X seg. base.
These things premised, we will now set down the several Problems or cases.

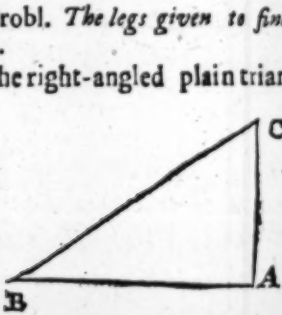
ſes in all plaine triangles right-angled and oblique, with the proportions by which they may be ſolved, & manner of ſolving them both by natural and Artificial numbers.

Of right angled Plain Triangles.

IN right angled plain triangles, the ſides comprehending the right-angle, we call the Legs, and the ſide ſubtending the right angle, we call the Hypotenuſe.

1 Probl. *The legs given to find an Angle.*

In the right-angled plain triangle,



A B C, the angle ACB is inquired.

The

The given legs { AB. 230 / AC. 143·72

AC. Rad : : AB. t A CB by the 1 Axiome.

That the quantity of this angle, or any other term required may be expressed in numbers, if the solution be to be made in natural numbers, multiply the second term given by the third, and their product divide by the first, the Quotient is the fourth proportional sought.

But if the solution be to be made in artificial numbers, from the sum of the Logarithmes of the second and third terms given, subtract the Logarithme of the first, the remainer shall be the Logarithme of the fourth proportional required.

Illustration by natural numbers.

As the Leg AC	143·72
Is to the Radius AC	10000000
So is the Leg. AB	235
To the tang. of ACB grad.	58.55

The product of the second and third

termes is 2350000000, which being divided by 143.72 the first term given, the quotient is 16351238 the tangent of the angle ACB which being sought in the Table the neereſt leſs is 16350527, and the arch anſwering thereto is Grad. 58. 55 parts.

Illuſtration by Logarithmes.

	Logarithmes.
As the leg AC 143.72	2.157517
Is to the rad. IC	10.0000
So is the leg AB 235	2.371068
To the tang. of C. gr. 58.55	10.213551

2 Probl. *The angles and one leg given, to find the other leg.*

In the right angled plain Triangle ABC, the leg AC is inquired:

The given { Angle ABC / Leg. AB

Rad. *AB*. : : t ABC. AC. by the first Axiome.

3 Probl.

3 Probl. *The Hypotennſe and a leg given to find an angle.*

In the right angled plain Triangle ABC, the angle ACB is inquired.

The given { Hypoth. BC. / Leg AB.

BC. Rad : : AB. s. ACB. by 2 Ax.

4 Probl. *The Hypotennſe and angles given, to find either leg.*

In the right angled plain Triangle ABC, the leg. AB is inquired :

The given { Hypot. BC. / Angle ACB.

Rad. BC : : s. ACB. AB. by 2 Ax.

5 Probl. *The angles and a leg given, to find the Hypotennſe.*

In the right-angled plain Triangle ABC, the Hypotenuſe BC is inquired;

The given { Angle ABC. / Leg AC.

s. ABC. AC : : Rad. BC. by the ſecond Axiome.

6 Probl.

6 Probl. *The Hypotenuse and leg given, to find the other leg.*

In the right-angled plain Triangle *ABC*, the leg AB is inquired,

The given $\begin{cases} \text{Hypot. BC} \\ \text{Leg. AC} \end{cases}$

1. BC. Rad :: AC. s. A B C, by the 3 Problem.
2. t. A B C. *A C* :: Rad. *A* B. by the 2 Probl.

7 Probl. *The legs given, to find the Hypotenuse.*

In the right-angled plain Triangle ABC the Hypotenuse BC is inquired.

The given legs $\begin{cases} AB \\ AC \end{cases}$

1 AB. Rad :: A C. t. *A* B C by the 1 Problem.
2 s. *A* B C. *A C* :: Rad. B C. by the 5 Problem.

Of Oblique angled plaine Triangles.

1 Probl. *Two sides and an angle opposite to one of them given, to find the angle opposite to the other side.*

In the Oblique angled plain Triangle DCB the angle CBD is inquired.

The given { Sides { DC 865 / CB 632 } / Angle CDB 26.37 }

But here it must be known whether the angle sought, be acute or obtuse. CB. s CDB :: CD, s. CBD. by the second Axiome.

IIn-

Illustration by natural Numbers.

As the side CB		632
Is to the sine of CDB	26.37	4441651
So is the side C D		865
To the sine of CBD	37.43	6079172

For if you multiply 4441661 by 865, the product will be 3842036765, which product being divided by the first term 632, the quotient 6079172 is the sine of 37 deg. 43 parts.

Illustration by Artificial numbers.

As the side CB	632	2.800717
Is to the sine of D	26.37	9.647545
So is the sides CD	865	2.937016
		12.584561
To the sine of B	37.43	9.783844

2 Probl. *Two sides with the angle comprehended by them given, to find either of the other angles.*

In the Oblique angled plain Triangle BDC.

The angle DBC is inquired.

The

The given { Sides { DC, BC; Angle DCB }

Subtract the angle given, out of 180d. the remainer is the sum of the other angles, and

½ Z cur. DC and BC. ½ X cur :: t ½ Z ang. t ½ X ang. *by the* 3 *Ax.*

To the half sum of the other angles, adde the half differenee found, and you will have the greater angle, subduct it & you will have the lesser, in our example ½ Z ang. + ½ X ang. is = DBC sought, because that is the greater ang.

3 Probl. *The angles and a side given, to find either of the other sides.*

In the Oblique angled plain triangle CBD,

The side D*B* is inquired.

The given { Angles { DCB, C*D*B; Side CB }

s. CDB, CB :: s. *D*CB. DB. by the second Axiome.

4 Prob.

4 Probl. *Two sides with the angle comprehended by them given, to find the third side.*

In the oblique angled plain triangle DCB

The side CB is inquired.

The given { Sides { DB / CB ; Angle CBD }

First, find the angle BCD by the third Axiome.

Secondly, find the side CB by the second *Axiome*.

5 Probl. *The three sides given to find an angle.*

In the Oblique angled plain triangle *CBD*.

The angle *CDB* is inquired.

The three sides being given { *DC* / *CB* / *DB* }

The resolution of this Problem doth require two operations, first, for the segment of the base *GB*.

DB.Z crur.DC & CB :: X crur.GB. by the fourth axiome.

And DB—GB=DG, and ½DG =DA or AG.

And DC. Rad :: AD. s ACD, by the second axiome whose complement is ADC.

And AG + GB = AB, therefore the ABC may be also found in like manner.

CHAP. V.

Of the affections of Spherical Triangles.

HAving done with plain triangles, we come next to speak of Spherical.

1 A Spherical triangle is that which is described on the sur-face of the Sphere.

2 The sides of a Spherical Triangle are the arches of three great circles of the Sphere, mutually intersecting each other.

3 The

3 The measures of *S*pherical angles are the arches of great circles described from the angular point between the sides of the angles, those sides being continued to quadrants.

4 Those are said to be great circles which bi-sect the *S*phere.

5 Those circles which cut each other at right angles, the one of them passeth through the poles of the other and the contrary.

6 In every *S*pherical triangle, each side is less then a semicircle, for if in the triangle A B C, you produce the

sides A B, B C, till they meet in the point D the arches B A D, and B C D are each of them a semicircle because they intersect each other in the points B and D, and therefore the arch *B* A

or

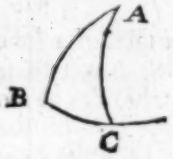
B C is leſs then a ſemicircle. In
ke manner if the ſides A B and A C
e produced, the ſide A C may be alſo
roved to be leſs then a Semicircle.

7 In every Spherical Triang. any two
ides are together greater then the
hird, for otherwiſe they cannot poſſi-
ly make a triangle.

8 The ſum of the ſides of a Sphe-
ical Triangle are leſs then two ſe-
icircles. For if any two ſides be pro-
uced as ſuppoſe A B, B C till they
oncur in the point D, the arches
B A D, B C D ſhall be each of them a
ſemicircle, but in the trian. A D C, the
ides A D and C D are together grea-
er then A C by the laſt aforegoing,
herefore the three ſides A B, B C, A C
re together leſs then the two ſemi-
ircles B A D, B C D,

9 If two ſides of a Spherical tri-
angle be equal to a ſemicircle, the two
angles at the baſe ſhall be equal to
wo right. If they be leſs then a ſe-
micircle, the two angles ſhall be leſs
then

then two right: but if greater then a ſemicircle, the two angles ſhall be greater then two right. As in the ſpherical Triangle ABC let the ſides AB, AC be equal to a Semicircle, then are the angles at B & C equal to two right, for the arch BC being produced to D, the angle ACD ſhall be equal to B, & ſeeing that the two angles at C are equal to two right, the two angles ACD and B ſhall be alſo equal to two right.



Again. Let the ſides AB and AC be leſs then a ſemicircle, ſeeing that the two angles at C are equal to two right, and the angle B leſs then the angle A C D, the angles A C B and B are together leſs then two right.

Laſtly, Let the ſides A B and A C be more then a ſemicircle, the angles at C being equal to a ſemicircle, and the angle at B greater then the angle

gle ACD, the angles ACB & B shall be greater then two right.

10 The sum of the three angles of a Spherical triangle are greater then two right angles and less then six.

Demonst. In the triangle ABC let the side BC be produced to D, then shall the angle ACD be either more or less or equal to the angle ABC; first, suppose them equal, then the arches AB and AC shall be equal to a semicircle, by the last aforegoing, and the angles ABC and ACB are equal to two right, and therefore the three angles A.B.C are more then two right.

Again. Let the angle ACD be less then the angle B, then the sum of the arches AB & AC shall be more then a Semicircle, and therefore the angles ABC and ACB greater also then two right

right, and therefore much more are the three angles A. B. C. greater then two right.

Lastly. Let the angle A C D be greater then the angle A B C, and make the angle D C E equal thereto, and the side AB being produced to E that the arch B E and C E may meet, and let the arch C A be produced to F, then shall the arches EB and EC be together equal to a semicircle, and therefore A E and E C are together less then a semicircle, and the angle E A F or B A C is greater then the angle A C E *by the ninth hereof*, but the angles A C E, A C B and B are equal to two right, therefore the angle A C B, A B C & B A C are greater then two right.

And because every angle of a Spherical Triangle is less then two right, the three angles together must needs be less then six, as was to be proved. *Therefore*,

11 Two angles of any Spherical Triangle are greater then the difference

rence between the third angle and a semicircle also.

12 Any side being continued, the exteriour angle is less then the two interiour opposite ones.

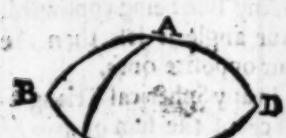
13 In any Spherical Triangle, the difference of the sum of two angles, and a whole circle is greater then the difference of the third angle, and a semicircle.

14 In any Spherical Triangle, one side being produced, if the other two sides be equal to a semicircle, the outward angle shall be equal to the inward opposite angle upon the side produced: if they be less then a semicircle, the outward angle shall be greater then the inward opposite angle: if greater then a semicircle, the outward angle shall be less then the inward opposite angle.

In the Spherical Triangle ABC let the side B C be continued to D, and let the sides AB and AC be together equal to a semicircle. I say then that the outward angle ACD is equal to

 the

the inward oppoſite angle at B, be-



cauſe the angle B = D and the angle A C D = D and therefore angle A C D = B.

Again, Let the ſides A B and A C be leſs then the ſemicircle B A D if the common arch *A* B be taken away, there ſhall remain the arch A C leſs then the arch A D and therefore the ang. A C D ſhall be greater then the angle *D*, therefore alſo more then B.

Laſtly, If the ſides A B and A C be together more then a ſemicircle, taking away the common arch A B, the remaining arch A C ſhall be greater then A D, and the angle A C D leſſer then D, and therefore alſo leſſer then B as was to be proved.

15 A Spherical Triangle is either right or angled or oblique.

16 A right angled Spherical Triangle is that which hath on right angle at the least.

17 The legs of a right angled spherical Triangle are of the same affection with their opposite angles.

In the spherical Triangle A B C right angled at A let the side AB be a quadrant, I say then that the angle ACB shall be a quadrant also, because B is then the pole of the arch AC, and the arch BC perpendicular thereunto, therefore also in the Triangle A C D, the side A D, being more then a quadrant, the angle A C D shall be also more then a quadrant; it being more then the right angle A C B, and in the right angles Spherical Triangle AEC the side AE being less then a quadrant,

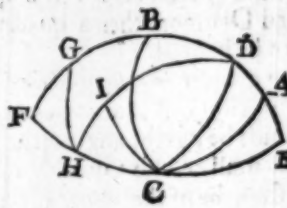
drant the angle A C E ſhall be alſo leſs then a quadrant, it being leſs then the right angle A C B.

18 In a right angled ſpherical Triangle, if either leg be a quadrant, the Hypotenuſa ſhall be alſo a quadrant; but if both the legs ſhall be of the ſame affection, the Hypotenuſe ſhall be leſs then a quadrant; if of different, then greater, and the contrary.

In the right angled triangle ABC right angled at A let the ſide A *B* be a quadrant, *I* ſay then that the Hypotenuſe *B C* is alſo a quadrant, becauſe the angle A *C* B is right, by the *laſt aforegoing*, and the arches A B and *B C* which are perpendicular to the arch A *C* doe meet in the pole B.

Again, Let the ſides *A* B and AC be continued to their oppoſite pole at F, then ſhall the Triangle F B *C* be equal to the triangle. A B *C*, but the arch G *H* being drawn by the points G and H, the baſe G H will be common to the right angled triangle A H G

A H G, whose legs A G and A H are greater then the quadrants A B & A C and also to the other right angled triangle F G H whose legs F G and F H are less then the quadrants F B & F C, and G H is less then the quadrant B C, which is the measure of the right an-



gles at F and A, if it be not less, it must be either more or equal to it, it cannot be more because the triangle A B C having all the angles right, can have no side greater then a quadrant, and it cannot be equal, because neither of the legs are a quadrant.

Lastly, In the triangle D A H right angled at A, the leg A D is less then the quadrant A B, and the leg H A

is greater then the quadrant A C, therefore the Hypotenusa D H is also greater then a quadrant, for A C and D C are each of them quadrants by the work, if therefore upon the pole D you describe the arch C I it will cut the Hypotenuse D H in the point I, and therefore D I is a quadrant and D H more then a quadrant, as was to be proved.

19 In a right angled spherical triangle, if either of the angles at the Hypotenusa be a right angle, the hypotenusa shall be also a quadrant, but if both shall be of the same affection it shall be less, if of different then greater and the contrary.

In the triangle A B C right angled at C if either of the angles at A or B be right, the side opposite thereto shal be also right *by the 17 hereof*, and the Hypotenusa A B shall be a quadrant by the last aforegoing, but if the angles at A and B be both acute or obtuse the sides A C and C B shall be both acute or obtuse also *by the 17th hereof*,

hereof, and the Hypotenuſe A B leſs then the quadrant by the laſt aforegoing: but if either of the angles at A & B be acute, and the other obtuſe, one of the legs ſhall be leſs, the other more then a quadrant, *by the 17th hereof*, and the Hypotenuſe *A* B more then a quadrant by the laſt aforegoing, as was to be proved. *Therefore*

20 In a right angled ſpherical Triangle either of the Oblique angles is greater then the complement of the other, but leſs then the difference of the ſame complement to a ſemicircle.

21 An Oblique angled ſpherical Triangle, is either acute or obtuſe.

22 An acute angled ſpherical Triangle hath all its angles acute.

23 An obtuſe angled ſpherical Triangle hath all its angles either obtuſe or mixt, *viz.* acute and obtuſe.

24 In any Spherical Triangle, whoſe angles are all acute, each ſide is leſs then a quadrant.

In the Triangle A B C let all the angles be acute, and A the greateſt

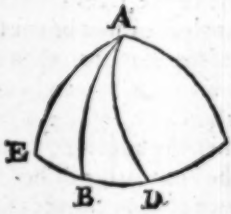
angle, then is B C greater then either of the other arches A B, A C, and the arch A E being drawn at right angles to the arch A B and made equal to A C, the arch B E is leſs then a quadrant, becauſe the legs of the right angled Triangle A B E, *viz.* A E

and A B are each of them leſs then a quadrant, and therefore the arch B C is much more leſs then a quadrant, and if B C the greateſt arch be leſs then a quadrant, the ſides A B & A C which are each of them leſs then the arch B C, muſt needs be leſs then quadrants alſo.

25 In any oblique angled ſpherical Triangle, if the angles at the baſe be of the ſame affection, the perpendicular

dicular drawn from the vertical angle shall fall within, if of different without.

In the oblique angled Triangle A B C whose angles at B & C are acute, the perpendicular A D shall fall within the triangle, for if it fall not within, it must be the same with one of the sides, or els fall without the Triangle if it be the same with either side, the angle at B or C must be right, which is contrary to the proposition, if it fall without the triangle, suppose at E the angle A E B shall be right,



but the angle A B E is obtuse, *viz.* the complement of the acute angle A B C

and therefore the side A E is greater then a quadrant, *by the 17th thereof*, and the angle A C E being acute A E shall be also less then a quadrant by the same *Theorem*, but that the same side should be both more and less then a quadrant is absurd in this case therefore the perpendicular shall fall within the triangle.

But in the Triangle A E B, obtuse-angled at B, acute at E, the perpendicular A D shall fall without the triangle, upon the side E B continued, or if otherwise, it must be the same with one of the sides, or fall within the triangle, it cannot be the same with either of the sides, for then the angle at B or E should be a right angle, and cannot fall within the triangle because then the angles at B and E must either be both acute or both obtuse as hath been already proved, if therefore the angles at the base be of different affection, the perpendicular shall fall without, as was to be proved.

Chap. VI.

Of the Calculation of Spherical Triangles.

IN Spherical Triangles there are 28 varieties or caſes, 16 in rectangular and 12 in oblique angular, whereof all the rectangular, and 10 of the oblique angular may be reſolved by theſe two Axiomes following.

1 Axiome.

In all Spherical rectangled Triangles, having the ſame acute angle at the baſe: *The ſines of the Hypotenuſes are proportional to the ſines of their perpendicular.*

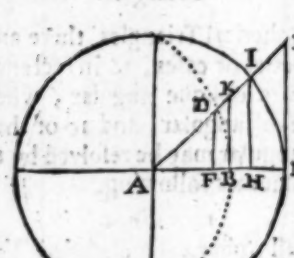
2 Axiome.

In all Spherical rectangled Triangles, having the ſame acute angle at the baſe: *The ſines of the baſes, and the tangents of the perpendiculars are proportional.*

Demonſt. Let A D B and A I M repreſent two Spherical Triangles ha-

wing the same angle at A, then is I H the sine of I M and F D the sine of D B. But

s. A D. A I Rad :: s D F s. I H.



Again. K B is the tang. of D B & L M the tang. of I M. And

s. A B. t. K B :: s. A M. t. L M. as was to be proved.

That all the cases of a right-angled Spherical Triangle may be resolved, by these two Axiomes, the several parts of the Spherical triangle, proposed must sometimes be continued to quadrants, that so the angles may be

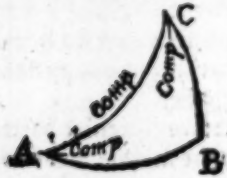
be turned into sides, the hypotenusaes into bases and perpendiculars, and the contrary, by which meanes, the proportions as to the parts of the Triangle given, instead of sines doe sometimes fall in Co-sines, and sometimes in co-tangents instead of tangents, which the L. *Nepair* observing, those parts of the right-ang. Spherical triangle which in such conversion doe for the most part change their proportion, he noteth with their complements, *viz.* the hypotenuse and both the acute angles, but the sides containing the right angle are not so noted; and these five he calleth the Circular parts of the triangle, amongst which the right angled is not reckoned, and therefore the two sides which doe contain it, are supposed to be joyned together.

Of these five Circular parts, one is alwayes in the middle, and two of the five are adjacent to that middle part, and the other two are disjunct, the parts adjacent, are called extreams ad-

adjacent, and the parts disjoyned, are called extreames disjunct.

Each of these five circular parts, may by supposition be made the midle part, and then the two circular parts which are next to that which is by supposition made the middle, are the extreames conjunct, the other two remote from the middle part assumed are the extream parts disjoyned.

As in the Triangle ABC, if comp. AC be made the middle part, comp. A, and comp. C are the extreames conjunct, and the sides AB and BC are the extreames disjunct, and so of the rest as in the Table following.



cMiddle

Mid. part	*Extr. conj.*	*Extr. disj.*
Leg. AB.	*Comp. A* *Leg BC*	*Comp. AC* *Com. C*
Comp. A	*Comp. AC* *Leg. AB*	*Com. C* *Leg. BC*
Comp. AC	*Comp. A* *Comp. C*	*Leg. AB* *Leg. BC*
Comp. C	*Comp. AC* *Leg. BC*	*Comp. AB* *Leg. AB*
Leg. BC	*Comp. C* *Leg. AB*	*Comp. A* *Comp. AC*

The parts of a right angled Spherical Triangle, being thus distinguished into five circular parts, (besides the right angle) the three parts remote from the right angle, being noted by their complements, and of these five one accounted the middle, the other either extreams adjacent or disjunct, the L. *Nepeir* as a consectary from the two preceding axiomes, for the help of memory, and therefore the more easie resolving of all spherical Triangles hath

hath composed this Catholick and Universal Proposition.

The Sine of the Middle part and Radius are reciprocally proportional, with the tangents of the extreams conjunct, and the co-sines of the extreams disjunct.

That is, *As Radius, to the tangent of one of the extreames conjoyned: So is the tangent of the other extream conjunct to the Sine of the Middle part.*

And also, *As Radius, to the co-sine of one of the extreames disjunct: So is the co-sine of the other extream disjunct, to the sine of the Middle part.*

Therefore, if the Middle part be sought, the radius must be in the first place, If either of the extreams, the other extream must be in the first place.

Onely note, That if the Middle part, or either of the extreams conjunct, be noted with its complement in the circular parts of the Triangle, instead of the sine or tangent, you must use the co-sine or co-tangent of such circular part or parts.

If

If either of the extreams disjunct, be noted by its complement in the circular parts of the triangle, instead of the co-sine, you must use the sine of such extream disjunct.

That these directions may be the better conceived, we have in the table following set down the circular parts of a Triangle under their respective titles, whether they be taken for the Middle part, or for the extreams, whether conjunct or disjunct, & unto those parts we have perfixed the sine or co-sine, the tangent or co-tangent as it ought to be by the former Rules.

Middle part	*Extr. conjunct*	*Extr. disjunct.*
Sine A B	*Co-tang A* *Tang. BC*	*Sine A C* *Sine C*
Cosine A	*Co-tang. A C* *Tang. A B*	*Sine C* *Co-sine BC*
Co-sine A C	*Co-tang. A* *Co tang. C*	*Co-sine AB* *Co-sine BC*
Co-sine C	*Co-tang. AC* *Tang. BC*	*Sine A* *Co-sine AB*
Sine BC	*Co-tang. C* *Tang. AB*	*Sine A* *Sine AC*

These

These things premised, we will now set down the several cases with the Analogies by which they may be solved, according to this Catholick proposition of right angled Spherical triangles first, and then of oblique, and in every case in which we deeme it necessary, we will Demonstrate the truth of the Proposition *by the first & second Axiomes of this chapter*, as where the middle part, is either one of the legs containing the right angle, or one of the oblique angles or the Hypotenusal.

Of right angled Spherical Triangles.

Problem. 1.

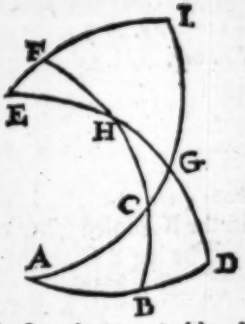
A leg with an angle opposite thereunto being given, to find the other leg. If it be known whether the Hypoteuuse or the other angle be more or less then a quadrant.

In the right angled Spherical Triangle A B C.

The leg A B or middle part is inquired

The given { Leg B C / ang. com. A } are extream conjunct.

Analogie Rad. cot. ang. A :: tang. B C. sine A B. *De-*



Demonst.
The Trian-
gles A G D
and A C B
have the
same acute
angle at the
base.
Therefore
tan.DG.AD
: : tang. B C
sine A B *by the second Ax. And by the 19th of the third.*

Rad. A D. co-tang. DG tang. DG. AD
therefore,
Rad. cot. DG : : tang. BC. sine AB
and D G is the measure of the angle, at A therefore, as Rad. &c.

Illustration by Natural numbers.

As the Radius.	10000000
To Co-tang. A grad. 30.	17320508
So tang. BC. grad. 22.89	4222108
To sine AB. 46 *d.* 994	7312904

Illustration by Artificial numbers.

As the Radius 90 *d.*	10.000000
To co-tang. A 30	10.238561
So tang. BC 22.89	9.625529
To sine AB. 46.994	9.864090

Problem 2.

A leg and an angle conterminate therewith being given, to find the other leg

In the R. angled spherical trian. ABC.

The leg B C one of the extreames conjunct is inquired.

The given

Angl. Comp. A the other extr. conj.

Leg AB the middle part.

Cot. A. Rad:: sine A B. tang. B C by the analogie in the first Problem.

Problem 3.

The legs given to find an angle.

In the right angled Spherical Triangle A B C.

The angle comp. A one of the extreams conjunct is inquired.

The given legs { AB the middle part. BC the other extrea. conjunct.

Analogie

Tang. B C. sine A B :: Rad. cot. A.

The

The inverse of that analogie in the
ſt Probleme.

Problem 4.

*The Hypotenuſe and a leg given, to find the
gle contained by them.*

In the right angled Spherical Tri-
gle, A B C.

The angle, comp. C, or middle part
inquired.

The given
ypot. comp. A C } the adjac. extr.
Leg B C }

Analogie, rad. cotang. A C : : tang.
C, coſine C.

Demonſt. The triangles E F H and
I G have the ſame acute angle at
he baſe, therefore,

I G = A C. Rad. I E : : t. F H = B C.
ine F E. *by the 2 Ax.*

Rad. co-tang. I G : : tang. I G. Rad.
y the 29 of the third. Therefore

Rad. cot. I G : : t. F H ſine F E,
or Rad. cot. A C : : t. B C. co-ſine C,
ecauſe F I the complement of F E is
he meaſure of A C B.

Pro

Problem 5.

A leg and an angle conterminate with it given to find the Hypotenuse.

In the right angled Spherical Triangle A B C.

The Hypot. comp. AC. one of the extreames conjunct is inquired.

The given

Leg BC. the other extr. conjunct.

Angle comp. C, the middle part.

Analogie. tang. BC. cosine C : : rad. cot. A C, it being the inverse of the Analogie in the former Problem.

Problem 6

The Hypotenuse and angle given, to find the leg conterminate with the given angle.

In the right angled Spherical Triangle, AB C.

The leg BC one of the extreames adjacent is inquired.

The given

Hypot. comp. AC the other extr. conj.

Angle comp. C. the middle part.

Analogie, cotang. A C, Rad : : cos. C. tang. *B* C.

This

This also is the inverse of the 3 Probleme, and therefore needs no further demonstration.

Problem 7.

The Oblique angles given, to find the Hypotenuse.

In the right angled Spherical Triangle A B C.

The Hypot. comp. AC, the middle part is inquired.

The given { Comp. A } The extreams
ing. { Comp. C } conjunct.

Analogie, Rad. cotang. C :: cotang. A cosine A C.

Demonst. The triangles C G H & C I F have the same acute angle at the base, C G is the complement of A C, H G the complement of G D the measure of A, and F I the measure of C, but

Tang. F I. C I :: tang. G H. sine G C by 2 axiom. therefore,

Tang. C. Rad. :: cot. A. cos. AC.

And as Rad. cot. C :: tang. C. Rad.

Therefore Rad. cot. C :: cot. A. cosine A C.

Pro-

Problem 8.

The Hypotenuse and an angle given, to find the other angle.

In the right angled Spherical Trigle A B C.

The angle comp. C, one of the extreams conjunct is inquired.

The given

Hypot. comp. AC, the middle part.
Ang, comp. A the other extr. conj.
Cotang. A. cosine AC : : Rad. cot. C,
by the Analogie in the preceding Probl.

Problem 9.

The Hypotenuse, and an oblique angle given, to find the leg opposite to the given angle.

In the right angled Spherical Triangle A B C.

The leg B C the middle part is inquired.

The { Hyp. comp. AC } extream
given { Angle comp. A } disjunct.

Analogie, Rad. sine A : : sine AC sine B C.

De-

Demonst. The Triangles A C B & A G D have the same acute angle at the base, therefore *by the first axiome*, AG. s. G D : : sine A C. sine B C.

Problem 10.

A leg and an angle opposite thereunto, being given, to find the Hypotenuse. If it be known whether it, or the other leg or angle be acute or obtuse.

In the right angled *Spherical* Triangle A B C.

The Hypotenuse comp. AC one of the extreams disjunct is inquired.

The { Ang.com. A the other extr. d.
given { Leg. BC the middle part.

Anal. Sine A. R. : : sine BC. sine C, *by the analogie in the preceding Probl.*

Problem 11.

The Hypotenuse and a leg given, to find the angle opposite to the given leg.

In the right angled *Spherical* Triangle A B C.

The angle comp. A one of the extreams disjunct is inquired.

The given
Hyp. comp. AC, the other extr. disj.
Leg. BC the middle part.

Anal. Sine AC, sine BC : : R. sine A *by the analogie in the* 10. *Probl.*

Problem 12.

An angle and leg conterminate with it given, to find the other angle.

In the right angled Spherical Triangle A B C.

The angle comp. A the middle part is inquired.

The { Angle comp. C } Extreames
given { Leg. BC } disjunct.

Anal. Rad. sine C : : cs BC. cos. A.

Demonst. The triangles C F I and C H G have the same acute angle at the base, therefore by the *first axiom*, Rad. FC. sine FI : : sine HC. sine HG, and the compl. of H C is the leg BC, & G D the measure of the angle at A is the complement of H G.

Therefore,

R. sine C : : co-sine B C. co-sine A.

Pro-

Problem 13.

An angle and leg opposite thereunto being given, to find the other angle. If it be known whether it, the Hypotenuse or the other leg be acute or obtuse.

In the right angled Spherical Triangle A B C.

The ang. comp. C one of the extreams disjunct is inquired.

The given

Leg B C the other extream disjunct.
Angle, comp. *A* the middle part.

Anal. cs B C. cs A :: Rad. sine C. *by the analogie in the preceding Problem.*

Problem 14.

The oblique angles given, to find either leg.

In the right angled Spherical Triangle A B C.

The leg B C one of the extreams disjunct is inquired.

The given

Angle, comp. C the other extr. disj.
Angle, comp. A, the middle part.

Anal. Sine C Rad : : cs A. cs BC. the inverse of that analogie *in the last Problem.*

Problem 15.

The legs given, to find the Hypotenuse.

In the right angled Spherical Tri^gle A B C.

The Hypotenuse comp. A C the middle part is inquired.

The given legs { A B } Extream
{ B C } disjunct.

Anal. Rad. cs AB : : cs BC cs AC.

Demonst. The triangles H B D and H C G have the same acute angle at the base, therefore

Rad. HB. sine BD : : sine CH. s CG,
But B A is the complement of B C, and B C the complement of C H, and C G the complement of AC.
Therefore,
Rad. cs A B : : cs B C. cs A C.

Pro-

Problem 16.

The Hypotenuse and a leg given, to find the other leg.

In the right angled Spherical Triangle A B C.

The leg A B one of the extreames disjunct is inquired.

The given

Leg B C the other extream dijunct.

Hypot. comp. A C the middle part.

Anal. cs B C. cs A C : : R. cs A B, *by the analogie in the* 15*th Problems.*

Chap. VII.

Of Oblique angled Spherical Triangles.

IN Oblique angled Spherical triangles, there are 12 Cases, 10 whereof may be resolved by the Catholick proposition, if the Spherical Triangle propounded be first converted into two Right, by letting fall of a per-

pendicular, sometimes within, sometimes without the triangle, if the angles at the base be both acute, or both obtuse, it falleth within the triangle, but if one of the angles at the base be acute, and the other obtuse, it falleth without, *by the 25 of the 5th. chapter:* however it falleth it must be alwayes opposite to a known angle, for your better direction, take this general Rule.

From the end of a side given, being adjacent to an angle given, let fall the perpendicular,

As in the triangle A B C, if there were given the side A B and the angle at *A*: by this rule the perpendicular must fall from B upon the side A C.

But if there were given the side AC, and the angle at *A*, the perpendicular must fall from C upon the side A B continued if need require.

And to know whether the side upon which the perpendicular shall fall, must be continued or not, that is to know whether the perpendicular shall

shall fall within or without the triang. If the former direction be not sufficient, the calculation will determine it, for if the arch found at the first operation (whether side or angle) be more then the arch given, the perpendicular shall fall without, if less, within the triangle, as shall plainly appear by our explanation of the ensuing Problemes.

Problem 1.

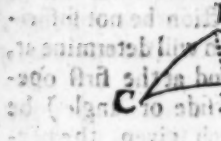
Two sides with an angle opposite to one of them being given, to find the angle opposite to the other. If it be known whether the angle sought be acute or obtuse.

In the oblique angled Spherical triangle *A* B C.

The angle B *A* C is inquired.

The given { Sides { A B 42. 15 / B C 29. 83 / Angle *A* C B 36. 14

The *Anal.* s. BA. s. BC : : s. C. s A.



For by the Catholick proposition.

1 R. s A B :: s A. s B D.
2 R. s B C :: s C. s B D.
Therefore, s A B. s B C :: s C. s A.

Illustration by natural numbers.

As the sine of AB 42.15	6710738
To the sine of BC 29.83	4974282
So is the sine of ACB 36.14	5897603
To the sine of BAC 25.92	4371552

Illustration by artificial numbers.

As the sine of AB 42.15	9.826770
To the sine of BC 29.83	9.696730
So the sine of ABC 36.14	9.770675
	19.467405
To the sine of BAC 25.92	9.640535

Pro-

Problem 2

Two angles with a side opposite to one of them being given, to find the side opposite to the other. If it be known whether the side sought be more or less then a quadrant.

In the Oblique angled Spherical Triangle A B C.

The side A B is inquired.

The given { Angles { BAC, ACB
Side BC

The Analogie is

s A. s C : : s BC. s AB, *by the last aforegoing.*

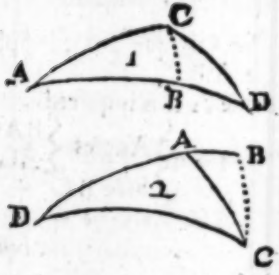
Problem 3.

Two sides and their contained angle given, to find the third side.

In the Oblique angled Spherical Triangle A C D, the side D C is inquired.

The given { Sides { AD, AC
Angle DAC

In this case the perpendicular may fall from the extremity of either side, but opposite to the angle given.



The terms of proportion.

1 ct AC. R : : cs DAC. t AB
And AD — AB = BD in the 1 Triang.
But AD + AB = BD in the 2 Triang.
2 cs AB. cs AC : : R. cs BC
3 R cs BC : : cs BD. cs DC

Therefore

cs AB. cs AC : : cs BD. cs DC.

Pro-

Problem 4.

Two sides with an angle opposite to one of them, to find the third side. If it be known whether the side sought, or the angle opposite to the other given side, be acute or obtuse.

In the Oblique angled Spherical Triangle A C D the side A D is inquired.

The given { Sides { A C / D C ; Angle A D C }

In this case let the perpendicular fall from the concourse of the given sides on the side inquired, continued if need be.

The terms of proportion.

1 ct C D. R. :: cs A D C. t. B D.
2 cs B D. cs C D :: R cs C B
3 cs C B. R :: cs A C. cs A B

Therefore

cs C D. cs B D :: cs A C. cs A B.
And B D + A B = A D in the 1 Triang.
But B D — A B = A D in the 2 Triang.

Problem 5.

Two sides and their contained angle given to find one of the other angles.

In the Oblique angled Spherical Triangle A C D, the angle ADC is inquired.

The given { Sides { AC / AD } / Angle CAD }

In this case the perpendicular may fall from the extremity of either side given opposite to the angle given.

The terms of proportion.

1. ct A C. R :: cs D A C. t AB.
And AD — AB = BD in the 1 Triang.
But AD + AB = BD in the 2 Triang.
ct C A D. s AB :: R. t BC.
t B C. R :: s BD. ct ADC

Therefore,

s A B. ct C A D :: s B D. ct A D C.

Problem 6

Two angles and a side opposite to one of them being given, to find the side between them. If it be known whether the side sought or

or the side opposite to the other given angle be acute or obtuse.

In the Oblique angled Spherical Triangle A C D, the side A D is inquired.

The given { Ang. { A D C / C A D ; Side A C }

In this case let the perpendicular fall from the extremity of the given side on the side inquired, continued if need be.

The terms of proportion.

1 ct A C. R :: cs D A C. t A B.
2 ct C A D. s A B :: R. t B C
3 R. t B C :: ct A D C. s B D.

Therefore,

ct C A D. s A B :: ct A D C. s B D.
And A B + B D = A D in the 1 Triang.
But D B — A B = A D in the 2 Triang.

Problem 7.

Two angles and a side opposite to one of them being given, to find the third angle. If it be known whether the angle inquired or the side oppo-

oppoſite to the other given angle be acute or obtuſe.

In the Oblique angled Spherical Triangle A C D, angle ACD is inquired.

The given { Angles { DAC / ADC ; Side AC }

In this caſe let the perpendicular fall from the angle inquired.

The terms of proportion.

1 ct C A D. R :: cs A C. ct A C B
2 s ACB. cs CAB :: R. cs B C
3 cs B C. R :: cs B D C. s BCD.

Therefore.

cs CAB. s ACB :: cs BDC. s BCD.
And ACB + BCD = ACD in the firſt Triangle.
But B C D − A C B = ACD in the ſecond Triangle.

Problem 8.

Two angles and the ſide between them given, to find the third angle.

In the Oblique angled Spherical Trian-

Triangle A C D, the angle A D C is inquired.

The given { Angle { ACD / CAD ; Side AC }

In this case, let the perpendicular fall from the extremity of the given side, and opposite to the angle inquired.

The terms of proportion.

1 ct CAB. R :: cs AC. ct ACB.
And ACD — ACB = BCD in the first Triangle.
But ACD + ACB = BCD in the second Triangle.
2 s ACB. cs CAB :: R. cs BC
3 R. cs BC :: s BCD. cs CDB

Therefore

s ACB. cs CAB :: s BCD. cs CDB.

Problem 9.

Two sides with an angle opposite to one of them being given, to find their contained angle. If it be known whether the angle inquired or the

the angle opposite to the other given side be acute or obtuse.

In the Oblique angled Spherical Triangle ACD, the angle ACD is inquired.

The given $\left\{\begin{array}{l}\text{Sides}\left\{\begin{array}{l}AC\\CD\end{array}\right.\\\text{Angle } ADC\end{array}\right.$

In this case, let the perpendicular fall from the angle inquired.

The terms of proportion.

1 ct CDB. R :: cs CD. ct BCD.
2 ct CD. cs BCD :: $\frac{\text{R. t B C}}{}$
3 R. $\frac{\text{t B C}}{}$ :: ct AC. cs ACB

Therefore

ct CD. cs BCD :: ct AC. cs ACB.
And BCD + ACB = A C D in the first Triangle.
But BCD — ACB = ACD in the second Triangle.

Problem 10.

Two angles and the side between them given, to find either of the other sides.

In the Oblique angled Spherical Tri-

Triangle A C D, the ſide D C is inquired.

The given { Angles { D A C / A C D ; Side A C }

In this caſe, let the perpendicular fall from the concourſe of the ſide given and ſought, on the third ſide continued if need be.

The terms of proportion.

1 ct C A B. R :: cs A C. ct A C B.
And A C D — A C B = B C D in the firſt Triangle.
But A C D + A C B = B C D in the ſecond Triangle.
2 ct A C. cs A C B :: R. t B C
3 t B C. R :: cs B C D. ct C D

Therefore

cs B C A. ct A C :: cs B C D. ct C D.

Problem 11.

Three ſides being given to find an angle.

To reſolve this Problem there muſt be ſome preparation made, for that the

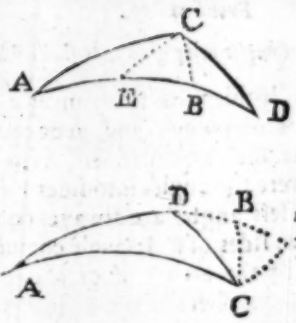
the Catholick proposition is not of it self sufficient either for this or for the ensuing Problem. And therefore the Lord *Nepier* to bring these Problems within some compass of the Catholick Proposition, first, findeth the difference of the segments of that side, which being made the base of the Triangle is divided into two parts by letting fall a perpendicular, and that by help of this *Analogie*, which we have demonstrated *in the second Book of our Trigonometria Britannica, chap. 2.*

As the tangent of half the base.
Is to the tangent of half the sum of the other sides.
So is the tangent of half the difference of those sides.
To the tangent of half the difference of the segments of the base.

In the Oblique angled spherical Triangle ACD, the angle CAD is inquired.

The given { Sides { AC, CD } Base AD }

The



The terms of proportion.

t $\frac{1}{2}$ AD. t $\frac{1}{2}$ Z cru. AC. CD :: t $\frac{1}{2}$ X cr. t $\frac{1}{2}$ AE

And $\begin{cases} \frac{1}{2}AD + \frac{1}{2}AE = AB \\ \frac{1}{2}AD - \frac{1}{2}AE = BD \end{cases}$

In the first Triangle.

$\frac{1}{2}AE + \frac{1}{2}AD = AB$

$\frac{1}{2}AE - \frac{1}{2}AD = BE$ or BD.

In the second Triangle.

Hence to find the angle at A.

R. ct AC :: t AB. cs CAB.

Pro-

Problem 12.

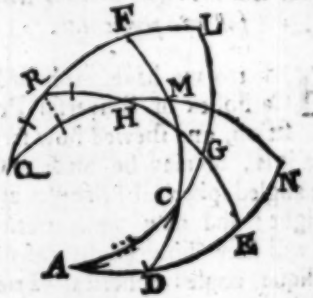
The three angles being given, to find a side.

This Problem is the converse of the last aforegoing, and to be resolved after the same manner, if so be we convert the angles into sides: for the two less angles are alwayes equal unto two sides of a Triangle comprehended by the arches of great circles drawn from there Poles, and the complement of the greatest angle to a semicircle must be taken for the third side. If in this triangle therefore thus connected, you shall by the preceding Problem find an angle, that angle so found shall be one of the three sides inquired.

As the Triangle ACD the poles of those arches are H. R and Q, which connected make the Triangle H R Q, the sides of the former Triangle being equal to the angles of the latter, taking for one of them, the complement of the greater angle to a Semicircle:

circle: as A D is equal to the angle at H, or the arch E N.

D C is equal to the angle at Q, or the arch M F.



And A C is equal to the complement of the angle H R Q, or the arch G L. Therefore if the angles, A, D, C, be given, the sides Q R, Q H. R H are likewise given.

If therefore we resolve the triangle H R Q by the directions of the last Problem, the angle so found, shall be the side inquired.

Chap.

Chap. VIII.

Of the Solution of Oblique angled Spherical Triangles without letting fall a perpendicular.

Hitherto we have explained the Catholick or Universal Proposition, and shewed how serviceable it is, or may be made in Oblique angled spherical Triangles, as well as right: and now for varietie sake we will shew how all the cases in an Oblique angled spherical Triangle may be resolved without letting fall a perpendicular, and that by help of these Propositons following.

Propos. 1.

In any spherical Triangle: *the sines of the angles are proportional to the sines of their opposite sides*, and the contrary.

This we have demonstrated from the Catholick Proposition, *in the first Problem of the last chapter.*

Propos.

Propos. 2.

In all Oblique angled Triangles whose sides together are less then a semicircle.

As the sine of halfe the sum of the angles at the base,
Is to the sine of the half difference of those angles,
So is the tangent of half the base,
To the tangent of half the difference of the sides.

And also:

As the co-sine of $\frac{1}{2}$ *the sum of the angles at the base*
Is to the co-sine of the halfe difference of the same angles,
So is the tangent of half the base:
To the tangent of the half sum of the sides.

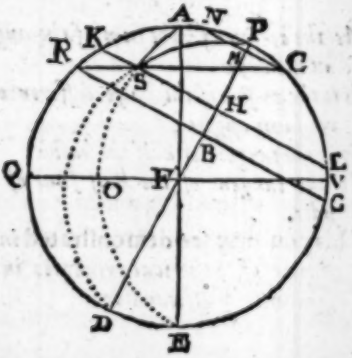
This you may see demonstrated *in the 2 chapter of Spherical triangles in our Trigonometria Britannica.*

Prop.

Propos. 3.

In all ſpherical Triangles: *As the difference of the verſed ſines of the ſum and difference of any two ſides (including an angle) is to the Diameter: So is the difference between the verſed ſine of the third ſide, & the verſed ſine of the difference of the other two ſides, to the verſed ſine of the angle comprehended of the ſaid two ſides.*

Let the ſides of the Triangle A P S



be known, and let the vertical angle be S A P, then shall A S the one side be equal to A R, and P R = Z tru. and P B the versed sine of P R, and P C is the difference of the sides A S and A P, and the versed sine of P C is P M, now then M B is the difference between B P the versed sine of P R the sum of the sides, and P M the versed sine of P C the difference of the sides.

M H is the difference between P H the versed sine of P S, and P M the versed sine of P C the difference of the sides, Q V is the diameter and O V the versed sine of P A S the angle sought: and the right lines N C. K L. and R G being parallel by the work, their intersegments M B and R C, M H and S C are proportional *by the 14 of the second chapter.*

M B. M H :: R C. S C
R C. S G :: Q V. O V

Therefore, M B. M H :: Q V. O V. as was to be proved.

Or thus, M B. M H :: ½ Q V. ½ O V.

Propos. 4.

In all spherical Triangles: *As the Rectangle of the sines of the sides containing the angle inquired, is to the square of Radius: So is the difference between the versed sine of the base and the versed sine of the difference of the other two sides, to the versed sine of the angle sought.*

Let the sides of the Triangle AEK be given and let the angle at A be inquired, and from O let fall the perpendicular O B, now then O E being

he difference of the sides A E & A K = A O the right line O Q is the ight sine thereof, and E Q the versed ine, in like manner, S M is the right ine, and E M the versed sine of E S, hat is. of the base E K, and M Q or O B is the difference of those versed ines, O K is the versed sine of the angle O A K in the measure of the parallel O F, and D X is the versed ine of the same angle in the measure of a great Circle, whose Diameter is D H, now then because of their like arches it shall be D C. D X :: L O. O K and because N E and O K are parallel, as also E C and O B, the angle B O K and C E N are equal, and the Triangles E C N, O K B like, and therefore the sides N E. E C. O B = M Q and O K. are proportional, and L O. D C :: O K D X } It will N E. E C :: O B. O K } be also. L O. N E. D C. E C :: O K. O B D X. O K. Or rejecting the comm. altitude O K, it will be

LO. NE. DC. EC :: OB. DX.
the verſed ſine of the angle ſought as
was to be proved.

Propoſ. 5.

In all Spherical Triangles: *As the Rectangle of the ſines of the ſides containing the angle inquired.*

Is to the ſquare of Radius,

So is the Rectangle of the ſines of the half ſums and half difference of the baſe, and difference of the legs.

To the Rectangle made of Rad. and half the verſed ſine of the angle inquired.

We have already proved, by the laſt afore-going.

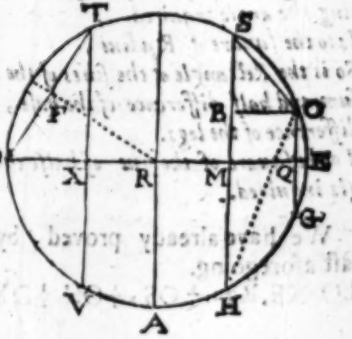
LO. NE. DC. AC :: OB. DX.

And therefore alſo

LO. NE. Rad. q :: $\frac{1}{2}$ OB. DX.

And in the following Diag. OEGH the ſum of ES and OE is the double meaſure of the angle BSO, the arch OS is the difference between the baſe ES and EO, the difference of the ſides

des AK and AE, and R. ½OS :: OH. ½OB, ½OS and ½OH = OB × R.



Therefore,

LO × NE. Rq :: ½OB × R. ½DX × R

And also,

LO × NE. Rq :: ½OS × ½OH ½DX × R which was to be proved.

Propos. 6.

In all Spherical Triangles: *As the Rectangle of the sines of the sides containing the angle inquired.*
Is to the square of Radius,
So is the Rectangle of the sines of the half sum and half difference of the base, and difference of the legs.
To the square of the sine of halfe the angle inquired.

We have already proved, by the last aforegoing.
LO×NE.Rq :: $\frac{1}{2}$ OS × $\frac{1}{2}$ OH. $\frac{1}{2}$ DX×R.

But the Rectangle made of Rad. & halfe the verſed ſine of an arch, is equal to the ſquare of the ſine of half the arch, as in the former figure, let the arch given be D T, then is D X the verſed ſine of that arch, and D F the right ſine of halfe the arch, and the Triangles D F R and D T X are like.

There-

Therefore,

DR. DF :: $\frac{1}{2}$DT (= DF.) $\frac{1}{2}$ DX and DR × $\frac{1}{2}$DX = DFq

Therefore,

IIO × NA. Rq :: $\frac{1}{2}$OS × $\frac{1}{2}$OH. DFq as was to be proved.

Problem 1.

Two sides with an angle opposite to one of them being given, to find the angle opposite to the other.

In the Oblique angled Spherical Triangle ABC, the angle BAC is inquired,

The given { Sides { AB, BC; Angle ACB }

The *Anal.* s BA. s BC :: s C. s A. *by the first Prop.*

Problem 2.

Two angles with a side opposite to one of them being given, to find the side opposite to the other.

In the Oblique angled Spherical Triangle A B C, the side A B is inquired.

The given $\begin{cases} \text{Angles} \begin{cases} \text{BAC} \\ \text{ACB} \end{cases} \\ \text{Side BC} \end{cases}$

The *Anal* s A. s C :: s BC. s AB *by the first Prop.*

Problem 3.

Two sides and their contained angle being given to find the other angles.

In the Oblique angled Spherical Triangle A B C.

The

The angles { BAC / ACB } are inquired.

The given { Sides { AB / BC } / Angle ABC }

1 *Operation.*

s ½ Z crur. s ½ X crur. :: ¹
ct ½ B. t ¹ X ang. A and C.

2 *Operation.*

cs ½ Z crur. cs ½ X crur.
ct ½ B t ½ Z ang. A and C,
by the second Proposition of this chapter.
½ Z ang. + ½ X ang. = A C B,
½ Z ang. — ½ X ang. = B A C.

Problem 4.

Two angles and their contained side being given to find the other sides.

In the Oblique angled Spherical Triangle A B C.

The

The sides { AB, BC } are inquired.

The given { Angles { BAC, ACB } ; Side AC }

1 *Operation.*

s ½ Z ang. s ½ X ang.
t ½ A C. t ½ X crur. AB and BC.

2 *Operation.*

cs ½ Z ang. cs ½ X ang.
t ½ A C. t ½ Z cr. A B and B C.
by the second Proposition of this chapter.

½ Z crur. + ½ X crur. = A B.

½ Z crur. − ½ X crur. = B C.

Prob.

Problem 5

Two sides with an angle opposite to one of them to find the third side.

In the Oblique angled Spherical Triangle *A* B C, the side A C is inquired.

The given { Sides { A B / B C ; Angle A C B }

1 *Operation.*

s *A* B. s B C :: s C. s A. *by the first Proposition.*

2 *Operation.*

s ½ X ang. A and C. s ½ Z ang.
t ½ X crur. t ½ A C.

By the first operation of the last probl.
And ½ A C being doubled, giveth the side A C inquired.

Pro-

Problem 6.

Two angles with a side opposite to one of them being given, to find the third angle.

In the Oblique angled Spherical triangle ABC, the angle ABC is inquired.

The given { Angles { BAC, ACB ; Side AB }

1 *Operation.*

s C. s A :: s A B. s B C. *by the first Proposition.*

2. *Operation.*

s $\frac{1}{2}$ X crur. s $\frac{1}{2}$ Z crur.
t $\frac{1}{2}$ X ang. *A* and C. ct $\frac{1}{2}$ A B C.
by the 1 *oper. of the third probl.*
And the double of $\frac{1}{2}$ ABC is the angle A B C inquired.

Problem 7.

Two angles and a side opposite to one of them being given, to find the side between them.

In the oblique angled Spherical Triangle A B C, the side A C is inquired.

The given { Angles { B A C / A C B ; Side A B }

1 *Operation.*

s C. s A :: s A B. s B C. *by the first proposition.*

2 *Operation.*

s ; X ang. s ; Z ang.

t ; X crur. t ; A C.

By the 1 oper. of the 4 Problems.

And the double of ; A C, is the side A C inquired.

Pro-

Problem 8.

Two sides and an angle opposite to one of them being given, to find their contained angle.

In the Oblique angled Spherical Triangle A B C, the angle C B A is inquired.

The given { Sides { A B / B C. ; Angle A C B }

Operation.

s A B. s B C : : s C. s A. *by the* 1 *Prop.*

2 *Operation.*

$s\frac{1}{2}$ X crur. $s\frac{1}{2}$ Z crur.
$t\frac{1}{2}$ X ang. $ct\frac{1}{2}$ A B C,

by the 1 *oper. of the* 3 *Probl.*

And the double of $\frac{1}{2}$ A B C, is the angle A B C inquired.

Problem 9.

Two sides and their contained angle given, to find the third side.

In the Oblique angled Spherical Triangle A B C, the side A C is inquired.

The

The given $\begin{cases} \text{Sides} \begin{cases} AB \\ BC \end{cases} \\ \text{Angle } ABC \end{cases}$

First, find the other angles *by the third Probl.* Then you may find the third side *by the* 1 *Prop.*

Or thus:

Radq. s A B - s B C

s q ½ ABC. X vs AC & vs X crur. *by the* 4 *&* 6 *Prop. of this chap.*

And the verſed ſine X cr. + X vs AC

and vs X cr. = vs A C.

Problem 10.

Two angles and the ſide between them given, to find the third angle.

This Probleme is the converſe of the laſt, and to be reſolved in the ſame manner, either at 3 operations, finding firſt the other two ſides *by the* 4*th Probl.* and then the third angle, *by the* 1 *Propoſ.*

Or, els *by the* 4 & 6 *Propoſitions of this chap.* the angles being firſt converted into ſides, and their contained ſide into an angle as hath been ſhewed *in the* 12 *Probl. of the laſt chap.*

Pro-

Problem 11.

Three sides being given, to find an angle.

In the Oblique angled Spherical Triangle A B C, the angle C B A is inquired.

The sides AB. BC. & AC being given.

Analogie, s A B × s B C. Radq. s ½ Z A C and X cru. × s ½ X A C and X cru. sq ½ B. *by the 6 Propos.*

Problem 12.

Three angles being given, to find a side.

This is the converse of the last, and to be resolved after the same manner, the angles being first converted into sides, (as hath been already shewed in the 12 *Probl: of the last chapt.*) for then the angle so found, shall be the side inquired.

FINIS.